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On the Asymptotic Distribution of the Zeros of Hermite, Laguerre, and Jonquière Polynomials

WOLFGANG GAWRONSKI

Abteilung für Mathematik, Universität Trier, D 5500 Trier, West Germany

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0. INTRODUCTION AND SUMMARY

There is an extensive literature in the field of orthogonal polynomials which is particularly concerned with detailed investigation of zeros and related questions [e.g., 3, 11, 17]. Most results on the zeros of the classical orthogonal polynomials are local in nature, such as inequalities, asymptotic expansions, and monotonicity properties. The global behaviour of the zeros is given in terms of the asymptotic distribution functions is well-known in the case of orthogonal polynomials with an absolutely continuous weight function possessing an a.e. positive derivative on a *compact* interval, $[-1, 1]$ say. Then, projecting the zeros from $[-1, 1]$ on the upper half of the unit circle, the points obtained on that semicircle are equidistributed there [cf. 17, Theorem 12.7.2, p. 310; 3, Theorem 9.3, p. 134]. For a class of weight functions supported by an *unbounded* interval, the asymptotic zero distribution for the associated orthogonal polynomials has been determined and investigated in a series of recent papers [e.g., 8–10, 12, 16, 19, 20]. The proofs either require certain extremal principles from potential theory or are based on three-term recurrence relations and use suitable quadrature formulae. Most of these works cover the two prominent representatives of orthogonal polynomials on an unbounded interval: Hermite and Laguerre polynomials. The primary object of this paper consists in an alternative computation of the asymptotic distribution function of the zeros for the two particular cases just mentioned (Section 2). Our proofs are based on a continuity theorem for Stieltjes transforms of distribution functions. This approach is well-known in probability theory for proving convergence of distribution functions via some suitable functionals (Section 1). To be more precise, we denote by x_{nv},

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$\nu = 1, \ldots, n$, the zeros of the Hermite or Laguerre polynomial of degree n which are well-known to be real and simple. Thus we may assume them to be numbered according to

$$x_{n1} < x_{n2} < \cdots < x_{nn}, \qquad n \in \mathbb{N}. \tag{0.1}$$

Further, we put

$$N_n(\xi) := |\{\nu \in \{1, \ldots, n\} \,|\, x_{n\nu} \leqslant \xi\}|, \qquad \xi \in \mathbb{R}. \tag{0.2}$$

Then the central results of this paper are established in Section 2 and read as follows:

$$\lim_{n \to \infty} \frac{1}{n} N_n(4n\xi) = \frac{2}{\pi} \int_0^{\xi} t^{-1/2}(1-t)^{1/2}\, dt, \qquad 0 < \xi \leqslant 1, \tag{0.3}$$

in case of Laguerre polynomials (Theorem 1) and

$$\lim_{n \to \infty} \frac{1}{n} N_n(\sqrt{2n}\,\xi) = \frac{2}{\pi} \int_{-1}^{\xi} \sqrt{1-t^2}\, dt, \qquad -1 \leqslant \xi \leqslant 1, \tag{0.4}$$

for Hermite polynomials (Theorem 3), thereby showing that the exact distribution functions suitably normalized tend to certain beta distributions.

In Section 3 we deal with Jonquière polynomials, P_n say, which can be defined by the rational function

$$f_n(z) := \sum_{\nu=1}^{\infty} \nu^n z^\nu = \frac{P_n(z)}{(1-z)^{n+1}}, \qquad n \in \mathbb{N} \tag{0.5}$$

[cf. 15, Vol. I, problem 46, p. 7; 7], where P_n is a polynomial of degree n. This function and its various generalizations (see (0.5)′ below and Section 3) are of some significance in various branches of mathematics and physics, such as summability, analytic number theory and the theory of structure of polymers. It is known [7, 14] that all zeros of P_n are real, nonpositive, and simple. With the notations of (0.1) and (0.2) we prove (Theorem 5)

$$\lim_{n \to \infty} \frac{1}{n} N_n(\xi) = \int_{-\infty}^{\xi} \frac{dt}{(-t)(\pi^2 + \log^2(-t))}, \qquad -\infty < \xi \leqslant 0. \tag{0.6}$$

Moreover, this result is extended to the generalization

$$f_\kappa(z) := \sum_{\nu=1}^{\infty} \nu^\kappa z^\nu, \qquad \kappa > 0, \tag{0.5$'$}$$

of (0.5), which possesses an analytic continuation onto $\mathbb{C}_1^*$, where throughout the paper for real α we use the notation

$$\mathbb{C}_\alpha^* := \{z \in \mathbb{C} \mid \text{if } \operatorname{Re} z \geqslant \alpha, \text{ then } \operatorname{Im} z \neq 0\}. \tag{0.7}$$

1. Auxiliary Results

In this section we collect some auxiliary results which are basic for the technical treatments of the proofs. To this end, in view of the examples announced in the Introduction above we assume throughout this section that

$$Q_n(z) =: a_n \prod_{\nu=1}^{n} (z - z_{n\nu}), \qquad n \in \mathbb{N},\ a_n \neq 0, \tag{1.1}$$

defines a sequence of polynomials the zeros of which are located on the nonnegative real axis. Further we suppose them to be numbered according to

$$0 \leqslant z_{n1} \leqslant z_{n2} \leqslant \cdots \leqslant z_{nn}, \tag{1.2}$$

(not necessarily distinct) and we put

$$N_n(t) := |\{\nu \in \{1, ..., n\} \mid z_{n\nu} \leqslant t\}|, \qquad t \geqslant 0. \tag{1.3}$$

That is, $N_n(t)$ denotes the number of zeros of Q_n being located "between 0 and t." It is well-known from the theory of orthogonal polynomials [e.g., 17, Theorem 12.7.2, p. 310; 3, Sect. III.9; 15, Vol. I, problem 194, p. 77] that for orthogonal polynomials on a compact interval, p_n say, the limit distribution of the zeros is closely related to the asymptotic behaviour of $p_n(z)^{1/n}$ or equivalently of $(1/n)p_n'(z)/p_n(z)$ in some domain of the complex plane. Modifying this approach and writing the logarithmic derivative of Q_n as

$$\frac{1}{n}\frac{Q_n'(z)}{Q_n(z)} = \frac{1}{n}\sum_{\nu=1}^{n} \frac{1}{z - z_{n\nu}} = \int_0^\infty \frac{1}{z-t}\, d\left(\frac{1}{n} N_n(t)\right) \tag{1.4}$$

[see also 5, Ex. 6, p. 309] we are led to proving convergence of distribution functions concentrated on the positive real axis via its Stieltjes transforms.

For the sake of clarity we are slightly more general and consider the class M_+, say, consisting of all distribution functions being concentrated on $[0, \infty)$. Moreover, we assume each $F \in M_+$ to be normalized such that F is right continuous and we define its Stieltjes transform by

$$h(z) := \int_0^\infty \frac{dF(t)}{z-t}, \tag{1.5}$$

being holomorphic throughout $\mathbb{C}_0^*$. Due to the normalization F is uniquely determined by h [e.g., 21, p. 336]. Now we can state the following continuity theorem for Stieltjes transforms. For completeness we add a short proof.

LEMMA 1. *Suppose that $F_n \in M_+$ and h_n are the corresponding Stieltjes transforms, $n \in \mathbb{N}$.*

(i) *If*

$$\lim_{n \to \infty} h_n(z) = h(z) \tag{1.6}$$

for z in a neighbourhood of some $z_0 \in \mathbb{C}_0^*$, then (1.6) *holds throughout $\mathbb{C}_0^*$.*

(ii) *If, further,*

$$-xh(-x) \to 1 \qquad \text{as } 0 < x \to \infty, \tag{1.7}$$

then there exists a (unique) $F \in M_+$ the Stieltjes transform of which is h and

$$\lim_{n \to \infty} F_n(t) = F(t) \tag{1.8}$$

at all continuity points of F.

Proof. Part (i) is a simple consequence of Vitali's theorem [e.g., 18, p. 168]. To show part (ii) we use standard arguments from probability theory [e.g., 2, Chap. 8] for proving convergence of distribution functions by looking at certain functionals. Since the class of functions

$$\left\{ f \mid f(t) = \frac{1}{z-t},\ t \geq 0,\ z \in \mathbb{C}_0^* \right\}$$

is M_+-separating [2, p. 165], i.e., in (1.5) h determines F uniquely, it remains to show [2, Proposition 8.15, p. 165] that under (1.6) and the continuity condition (1.7) the sequence $\{F_n\}$ is masspreserving [2, Definition 8.9, p. 162]. To this end first we conclude from (1.5) that $(z = -x)$

$$1 + xh_n(-x) \geq \tfrac{1}{2}(1 - F_n(x)), \qquad x > 0, \tag{1.9}$$

since for any $F_n \in M_+$ we have $(\lim_{t \to \infty} F_n(t) = 1)$

$$\begin{aligned} 1 + xh_n(-x) &= 1 - x \int_0^\infty \frac{dF_n(t)}{x+t} = \int_0^\infty \frac{t}{x+t}\, dF_n(t) \geq \int_x^\infty \frac{t}{x+t}\, dF_n(t) \\ &\geq \frac{1}{2} \int_0^\infty dF_n(t) = \frac{1}{2}(1 - F_n(x)). \end{aligned}$$

Thus (1.9) combined with part *i* implies

$$\overline{\lim_{n\to\infty}}\,(1-F_n(x))\leqslant 1+xh(-x),\qquad x>0,$$

and further, by (1.7),

$$\lim_{x\to\infty}\,\overline{\lim_{n\to\infty}}\,(1-F_n(x))=0.$$

Now, as in [2, proof of Proposition 8.29, p. 172] it follows that $\{F_n\}$ is masspreserving and the proof is complete.
Putting

$$F_n(t):=\frac{1}{n}\,N_n(t),\qquad t\geqslant 0,$$

the following translation of Lemma 1 into terms of the polynomials Q_n is immediate.

LEMMA 2. *Suppose that* $\{Q_n\}$ *is a sequence of polynomials of degree* n *with real non negative zeros only* (*see* (1.1)–(1.3)).

(i) *If*

$$\lim_{n\to\infty}\frac{1}{n}\frac{Q_n'(z)}{Q_n(z)}=h(z) \tag{1.6$'$}$$

for z *in a neighbourhood of some* $z_0\in\mathbb{C}_0^*$, *then* (1.6) *holds throughout* $\mathbb{C}_0^*$.

(ii) *If, further,* (1.7) *holds, then there exists a* (*unique*) $F\in M_+$ *with* (1.5) *and*

$$\lim_{n\to\infty}\frac{1}{n}\,N_n(t)=F(t) \tag{1.8$'$}$$

at all continuity points of F.

Remarks. (i) If $0\leqslant t_1<t_2$ and n is large enough, then (1.8)$'$ gives an approximation for the number of zeros of Q_n in $(t_1, t_2]$ by $n(F(t_2)-F(t_1))$.

(ii) In many applications the limit in (1.6)$'$ can be determined and then an evaluation of F is reduced to the inversion of a Stieltjes transform (cf. [21, Chap. VIII, Sect. 7]).

Next, we turn to Laguerre and Hermite polynomials, where throughout we use definitions and notations given in [17, Chap. V]. The following modification of Perron's formula for Laguerre polynomials (cf. [17, Theorem 8.22.3, p. 199]) will be fundamental in computing the asymptotic distribution function of the zeros of Laguerre and Hermite polynomials.

LEMMA 3. *If $L_n^{(\alpha)}(z)$ denotes the Laguerre polynomial of degree n, $\alpha \in \mathbb{R}$, then as $n \to \infty$*

$$L_n^{(\alpha)}(4nz) = \frac{1}{\sqrt{2\pi n}} (2z + 2\sqrt{z^2 - z})^{-\alpha-1} \left(1 - \frac{1}{z}\right)^{-1/4} \times \{e^{2z+2\sqrt{z^2-z}} (1 - 2z + 2\sqrt{z^2 - z})\}^n \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \tag{1.10}$$

for z in the half-plane

$$H := \{z \in \mathbb{C} \mid \operatorname{Re} z < 0\}. \tag{1.11}$$

The function involved in the $\mathcal{O}$-term is holomorphic in H.

Remarks. (i) For the fractional powers t^β, $\beta \in \mathbb{R}$, we have to choose that branch which is real and positive if t is so.

(ii) Further, we observe that according to this choice of the power the z-plane with a cut along $[0, 1]$ is mapped by

$$\zeta_+ = 2z + 2\sqrt{z^2 - z} \quad \text{and} \quad \zeta_- = 2z - 2\sqrt{z^2 - z} \tag{1.12}$$

onto the interior and the exterior of the circle $\{\zeta \mid |\zeta - 1| = 1\}$ in the $\zeta -$plane, respectively.

(iii) The following proof shows that the set H for which (1.10) holds can be extended but for our purpose the present form of Lemma 3 is sufficient.

Proof of Lemma 3. We start with the representation [17, formula (5.4.8), p. 105]

$$L_n^{(\alpha)}(z) = \frac{e^z z^{-\alpha}}{2\pi i} \int_\gamma \frac{e^{-t} t^{n+\alpha}}{(t-z)^{n+1}} \, dt, \qquad z \neq 0,$$

γ being a contour enclosing $t = z$, but not $t = 0$. Next, we obtain

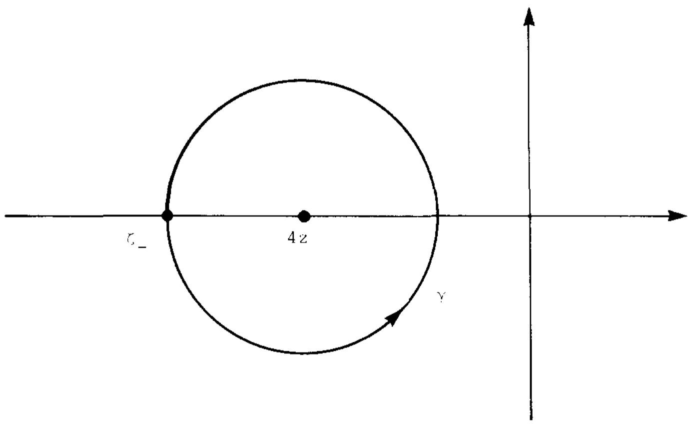
$$L_n^{(\alpha)}(4nz) = \frac{e^{4nz}(4z)^{-\alpha}}{2\pi i} \int_\gamma \left(\frac{e^{-\tau}\tau}{\tau - 4z}\right)^n \frac{\tau^\alpha}{\tau - 4z} \, dt, \qquad z \neq 0, \tag{1.13}$$

where now γ denotes a contour enclosing $\tau = 4z$, but not $\tau = 0$. For the asymptotic evaluation of (1.13) we employ the method of steepest descent for contour integrals (cf. [13, Chap. 4, Sect. 7]). To this end we put

$$p(\tau) := \tau - \log \frac{\tau}{\tau - 4z}, \qquad q(\tau) := \frac{\tau^\alpha}{\tau - 4z} \tag{1.14}$$

and suppose first that z is real and negative. The branch of the logarithm is real if τ is real and $\tau < 4z$. Possible saddle points for the integral in (1.13) are $\tau = \zeta_{\pm}$ as solutions of the equation $p'(\tau) = 0$. We choose $\tau = \zeta_-$ (located on the negative real axis in the τ-plane) and in (1.13), due to Cauchy's theorem,

$$\gamma := \{\tau = \tau(\phi) = 4z + (4z - \zeta_-)\, e^{i\phi} \mid 0 \leqslant \phi \leqslant 2\pi\}$$



Graph of γ.

(observe that $4z - \zeta_- = 2z + 2\sqrt{z^2 - z} = \zeta_+ > 0$ for $z < 0$, by (1.12)). Then the circle γ is contained in the half-plane $\{\tau \mid \operatorname{Re} \tau < 0\}$ and $\tau = \zeta_-$ is the only saddle point on γ (note that $\tau(\pi) = \zeta_-$). Further, we have

$$p''(\zeta_-) = \frac{1}{z} \sqrt{z^2 - z} \tag{1.15}$$

and for $\tau \in \gamma - \{\zeta_-\}$

$$\begin{aligned} \operatorname{Re}(p(\tau)) - p(\zeta_-) &= \operatorname{Re}(\tau - \zeta_-) + \log \left| \frac{\tau - 4z}{\zeta_- - 4z} \frac{\zeta_-}{\tau} \right| \\ &= \zeta_+ (1 + \cos \phi) + \log |\zeta_-/\tau| > 0, \end{aligned} \tag{1.16}$$

since $|\zeta_-| > |\tau|$. These observations imply that all assumptions of the method of steepest descent are satisfied [13, Theorem 7.1, p. 127] and we obtain from (1.13), (1.12), and (1.14)

$$
\begin{aligned}
L_n^{(\alpha)}(4nz) &= \frac{e^{4nz}(4z)^{-\alpha}}{2\pi i}\, 2e^{-np(\zeta_-)} \sqrt{\frac{\pi}{n}}\, \frac{q(\zeta_-)}{(2p''(\zeta_-))^{1/2}} \left(1+\mathcal{O}\left(\frac{1}{n}\right)\right) \\
&= \frac{1}{\sqrt{2\pi n}}\, e^{4nz} \left(\frac{\zeta_-}{4z}\right)^{\alpha} \left(\frac{e^{-\zeta_-}\zeta_-}{\zeta_- - 4z}\right)^{n} \frac{1}{\zeta_- - 4z}\, \frac{1}{i(p''(\zeta_-))^{1/2}} \left(1+\mathcal{O}\left(\frac{1}{n}\right)\right) \\
&= \frac{1}{\sqrt{2\pi n}}\, \zeta_+^{-\alpha-1} (e^{\zeta_+}(1-\zeta_-))^{n}\, \frac{1}{(-i)(p''(\zeta_-))^{1/2}} \left(1+\mathcal{O}\left(\frac{1}{n}\right)\right).
\end{aligned}
$$

In forming $(p''(\zeta_-))^{1/2}$ the branch of $\omega_0 = \arg(p''(\zeta_-))$ must satisfy $|\omega_0 + 3\pi| \leqslant \pi/2$, since

$$\lim_{\phi \to \pi + 0} \arg(\tau(\phi) - \zeta_-) = 3\pi/2; \text{ hence we have } \omega_0 = -3\pi.$$

Now (1.10) follows from (1.12) and (1.15) in case $z < 0$ (note that $-i = e^{3\pi i/2}$).

In order to extend the validity of (1.10) to the half-plane H, we have to pay attention to the key condition of the saddle point method, that is,

$$\operatorname{Re}(p(\tau) - p(\zeta_-)) > 0 \tag{1.17}$$

for $\tau \in \gamma - \{\zeta_-\}$. To this end we keep $4z \in H$ fixed in the τ-plane and construct $\gamma \subset \mathbb{C}_0^*$ in (1.13) such that

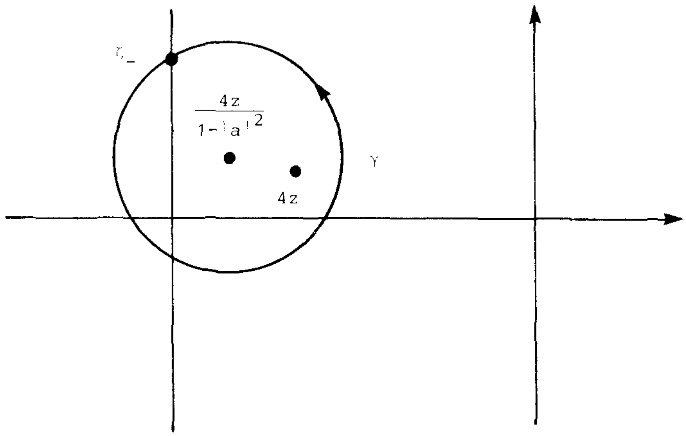
$$\operatorname{Re}(\tau - \zeta_-) > 0 \qquad \text{and} \qquad \left|\frac{\tau - 4z}{\zeta_- - 4z}\, \frac{\zeta_-}{\tau}\right| > 1 \tag{1.18}$$

for $\tau \in \gamma - \{\zeta_-\}$ (observe (1.16)). The second condition in (1.18) is equivalent to

$$\left|\tau - \frac{4z}{1-|a|^2}\right| > \frac{|4z|}{1-|a|^2}\,|a|$$

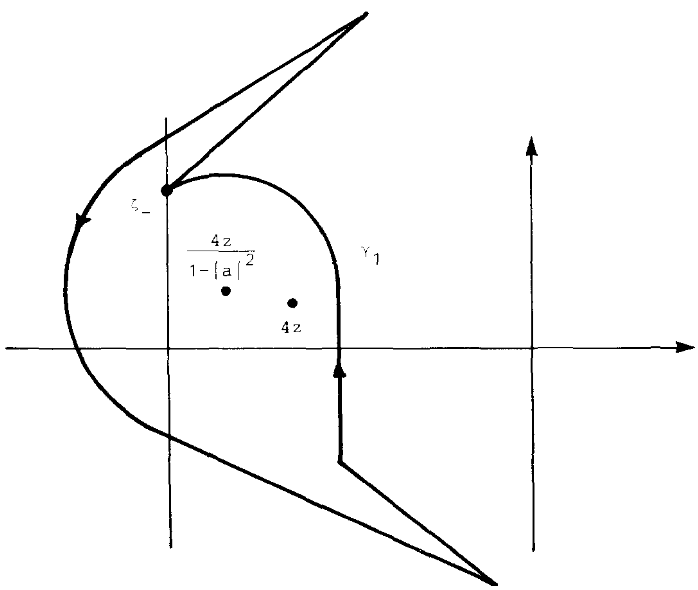
where $|a| = |\zeta_+ - 1| < 1$ for all $z \in \mathbb{C} - [0, 1]$ (compare Remark (ii) above). First in (1.13) we start with γ as the circle

$$\gamma := \left\{\tau = \tau(\phi) = \frac{4z}{1-|a|^2} + \frac{|4z|}{1-|a|^2}\,|a| e^{i\phi} \,\middle|\, 0 \leqslant \phi \leqslant 2\pi\right\}$$



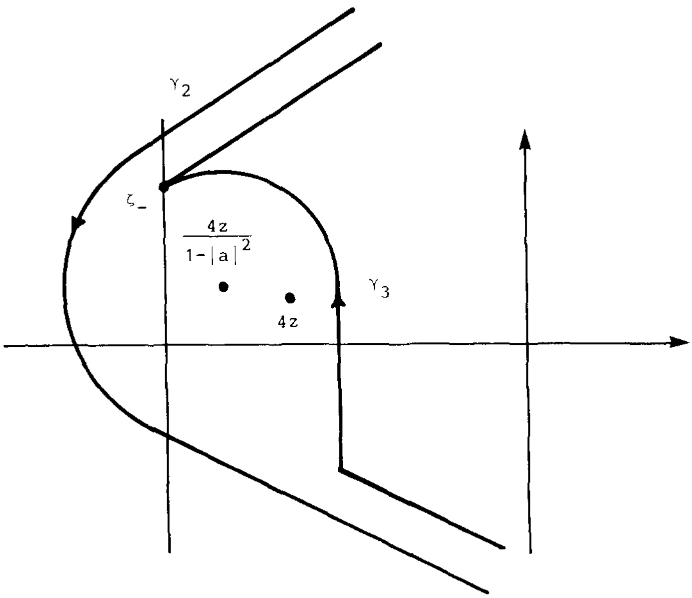
Graph of γ.

and observe that $\tau = 0$ and $\tau = 4z$ are in the exterior and the interior respectively. Further, γ passes through ζ_- and $\mathrm{Re}(4z - \zeta_-) = \mathrm{Re}(\zeta_+) > 0$. However, if $\mathrm{Im}\, z \neq 0$, then part of γ is contained in $\{\tau | \mathrm{Re}(\tau - \zeta_-) < 0\}$. Next, by deforming γ into γ_1



Graph of γ_1.

we further extend γ_1 into $\gamma_2 \cup \gamma_3$



Graph of $\gamma_2 \vee \gamma_3$.

where γ_2 and γ_3 approach infinity through the half-plane $\{\tau \mid \operatorname{Re} \tau > 0\}$ (note that the integrand in (1.13) decays exponentially). By Cauchy's theorem the contribution along γ_2 vanishes and $\gamma_3 \subset \mathbb{C}_0^*$ may be chosen according to (1.18). Now the arguments for $z \in H$ leading from (1.13) to (1.10) are the same as above and the proof of lemma 3 is complete.

2. Orthogonal Polynomials

In this section we apply the results of the preceding section to Hermite and Laguerre polynomials.

First we deal with Laguerre polynomials $L_n^{(\alpha)}$, $\alpha > -1$, [17, Chap. V]. Then all zeros of $L_n^{(\alpha)}$ are positive which we assume to be numbered by (0.1), and also with this notation and (0.2) we put

$$N_n(L_n^{(\alpha)}; \xi) := N_n(\xi), \qquad \xi \geqslant 0, \tag{2.1}$$

the number of zeros of $L_n^{(\alpha)}$ not exceeding ξ. Starting with (1.4), $Q_n = L_n^{(\alpha)}$, Perron's asymptotic formula for Laguerre polynomials in the complex domain does not produce the "correct weight" $1/n$ for each zero

[17, Theorem 8.22.3, p. 199]. However, looking at local results shows that the largest zero x_{nn} is "roughly" located near $4n$, as $n \to \infty$ [17, 6.32, pp. 131, 132]. This observation leads to considering

$$Q_n(z) := L_n^{(\alpha)}(4nz) \tag{2.2}$$

rather than $L_n^{(\alpha)}$ itself. Clearly we have

$$N_n(Q_n; \xi) = N_n(L_n^{(\alpha)}; 4n\xi), \qquad \xi \geqslant 0. \tag{2.3}$$

Now Lemma 3 shows that (2.3) is a proper scaling for obtaining a non trivial limit distribution. More precisely, from Lemma 3 we get that

$$\lim_{n \to \infty} \frac{1}{n} \frac{Q_n'(z)}{Q_n(z)} = h(z), \qquad z \in H, \tag{2.4}$$

where h is the logarithmic derivative of the function $\exp(2z + 2\sqrt{z^2 - z})(1 - 2z + 2\sqrt{z^2 - z})$ (observe that Q_n has no zeros in H and the remainder term in (1.10) is holomorphic in H). Next, a straightforward computation gives

$$h(z) = \frac{2}{z - \sqrt{z^2 - z}} \tag{2.5}$$

and it follows that the continuity condition (1.7) is satisfied. Hence (2.4) and Lemma 2 imply that there exists a unique $F \in M_+$ such that

$$h(z) = \int_0^\infty \frac{dF(t)}{z - t}, \qquad z \in \mathbb{C}_0^*. \tag{1.5}$$

Since, by (2.5), h is holomorphic in the cut plane $\mathbb{C} - [0, 1]$ and can be extended analytically beyond the cut we even have

$$h(z) = \int_0^1 \frac{f(t)}{z - t}\, dt, \qquad z \in \mathbb{C}_0^*, \tag{2.6}$$

where $f \in L(0, 1)$ is the derivative of F that is its corresponding density. Now from (2.6) with (2.5) we conclude by an inversion theorem for Stieltjes transforms [21, Theorem 7b, formula (5)] that

$$f(t) = \frac{2}{\pi} t^{-1/2}(1 - t)^{1/2}, \qquad 0 < t \leqslant 1. \tag{2.7}$$

Combining these results we have proved:

THEOREM 1. *If $L_n^{(\alpha)}(z)$ denotes the Laguerre polynomial of degree n, $\alpha > -1$, and $N_n(\xi)$ is the number of zeros of $L_n^{(\alpha)}$ not exceeding ξ, $\xi \geqslant 0$, then*

$$\lim_{n\to\infty} \frac{1}{n} N_n(4n\xi) = \frac{2}{\pi} \int_0^{\xi} t^{-1/2}(1-t)^{1/2}\, dt, \qquad 0 < \xi \leqslant 1. \tag{0.3}$$

Remark. Writing in (2.7)

$$f(t) = \frac{\Gamma(\frac{1}{2}+\frac{3}{2})}{\Gamma(\frac{1}{2})\,\Gamma(\frac{3}{2})}\, t^{(1/2)-1}\,(1-t)^{(3/2)-1}, \qquad 0 < t \leqslant 1,$$

we recognize F to be the beta distribution with parameters (1/2, 3/2) which also may be written in the form

$$F(\xi) = \frac{2}{\pi}(\sqrt{\xi(1-\xi)} + \arcsin\sqrt{\xi}), \qquad 0 \leqslant \xi \leqslant 1,$$

[6, Chap. 24].

Theorem 1 gives the asymptotic number of zeros in intervals of the form $(4n\xi_1, 4n\xi_2]$ that is in intervals the length of which tends to infinity with n (compare remark (i) after Lemma 2). In contrast to this result the question arises: "how many" zeros are located in a "fixed" interval, $(\xi_1, \xi_2]$ say? The precise answer is given by:

THEOREM 2. *Under the assumptions and with the notations of Theorem* 1 *we have*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} N_n(\xi) = \frac{2}{\pi}\sqrt{\xi}, \qquad \xi > 0. \tag{2.8}$$

Proof. This runs parallel to the classical proof for the limit distribution of the zeros of orthogonal polynomials on a compact interval (cf. [3, Sect. III.9]). Therefore we restrict our considerations to some essential steps. Starting from the above-mentioned Perron's formula [17, Theorem 8.22.3, p. 199] we get

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{\nu=1}^{n} \frac{1}{z - x_{n\nu}} = \frac{-1}{(-z)^{1/2}}, \qquad z \in \mathbb{C}_0^*.$$

Power series expansion at $z = -1$ and comparing coefficients gives

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{\nu=1}^{n} \frac{1}{(1+x_{n\nu})^{k+1}} = \binom{-1/2}{k}(-1)^k, \qquad k \in \mathbb{N}_0.$$

Since

$$\binom{-1/2}{k}(-1)^k=\binom{2k}{k}\frac{1}{4^k}=\frac{2}{\pi}\int_0^{\pi/2}(\cos t)^{2k}\,dt=\frac{1}{\pi}\int_0^1\frac{x^k}{\sqrt{x(1-x)}}\,dx$$

further we obtain by the approximation theorem of Weierstrass,

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\sum_{\nu=1}^{n}\frac{1}{1+x_{n\nu}}\,g\left(\frac{1}{1+x_{n\nu}}\right)=\frac{1}{\pi}\int_0^1\frac{g(x)}{\sqrt{x(1-x)}}\,dx$$

for every continuous function g on $[0,1]$. Using a continuous approximation to the function

$$\begin{aligned} g(x) &:= 0, && x<\frac{1}{1+\xi} \\ &:= 1/x, && \frac{1}{1+\xi}\leqslant x\leqslant 1; \end{aligned}$$

where $\xi>0$ is fixed, we end with

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}N_n(\xi)=\lim_{n\to\infty}\frac{1}{\sqrt{n}}\sum_{x_{n\nu}\leqslant\xi}1=\frac{1}{\pi}\int_{1/(1+\xi)}^1\frac{dx}{x^{3/2}(1-x)^{1/2}}=\frac{2}{\pi}\sqrt{\xi}$$

and (2.8) is established.

Next, we turn to Hermite polynomials H_n [17, Chap. V]. Basically we could proceed as in case of Laguerre polynomials by deriving an analogue to Lemma 3. However, we use the close relation between the two types of polynomials given by [17, formula (5.6.1), p. 106]

$$H_{2m}(z)=(-1)^m\,2^{2m}m!\,L_m^{(-1/2)}(z^2),\ H_{2m+1}(z)=(-1)^m\,2^{2m+1}m!\,zL_m^{(1/2)}(z^2)$$

from which the following relations are immediate consequences,

$$\begin{aligned} N_{2m}(H_{2m};\xi) &= m-N_m(L_m^{(-1/2)};\xi^2-0), && \xi<0, \\ &= m+N_m(L_m^{(-1/2)};\xi^2), && \xi\geqslant 0, \end{aligned} \tag{2.9}$$

$$\begin{aligned} N_{2m+1}(H_{2m+1};\xi) &= m-N_m(L_m^{(1/2)};\xi^2-0), && \xi<0, \\ &= m+1+N_m(L_m^{(1/2)};\xi^2), && \xi\geqslant 0, \end{aligned} \tag{2.10}$$

where $N_m(\xi-0)$ denotes the left side limit as customary (see (0.2)). Now a straightforward computation based on Theorem 1 yields the corresponding analogue for Hermite polynomials in:

THEOREM 3. *If H_n denotes the Hermite polynomial of degree n and $N_n(\xi)$ is the number of zeros of H_n not exceeding ξ, $\xi \in \mathbb{R}$, then*

$$\lim_{n \to \infty} \frac{1}{n} N_n(\sqrt{2n}\xi) = \frac{2}{\pi} \int_{-1}^{\xi} \sqrt{1-t^2}\, dt, \qquad -1 \leqslant \xi \leqslant 1. \tag{0.4}$$

Remark. The limit distribution in (0.4) again is a beta distribution, however, here with support $[-1, 1]$ and parameters $(\frac{3}{2}, \frac{3}{2})$ [6, Chap. 24] and it can be written in the form

$$\frac{1}{\pi}\left(\xi\sqrt{1-\xi^2} + \arcsin \xi + \frac{\pi}{2}\right), \qquad -1 \leqslant \xi \leqslant 1.$$

In this context it should be mentioned that this beta distribution also occurs as limit of random distribution functions of the eigenvalues in a sequence of certain random matrices [1, 22].

Finally we state the analogue of Theorem 2 for Hermite polynomials giving the asymptotic number of zeros in a "fixed" interval $(\xi_1, \xi_2]$. Since it is an immediate consequence of Theorem 2, (2.9), and (2.10) we omit its simple proof.

THEOREM 4. *Under the assumptions and with the notations of Theorem 3 we have*

$$\lim_{n \to \infty} \frac{1}{\sqrt{n}} (N_n(\xi_2) - N_n(\xi_1)) = \frac{\sqrt{2}}{\pi} (\xi_2 - \xi_1), \qquad \xi_1 < \xi_2. \tag{2.11}$$

The classical uniform distribution result for orthogonal polynomials on $[-1, 1]$ mentioned in the introduction could also be derived from Lemma 2.

3. JONQUIÈRE POLYNOMIALS

In this section we deal with Jonquière's function f_κ (see (0.5)′) and the associated polynomials P_n (see (0.5)). It is known (e.g., [14]) that f_κ has exactly n zeros in $\mathbb{C}_1^*$, if $n-1 < \kappa \leqslant n$, $n \in \mathbb{N}$, all of them are located on the nonpositive real axis and they are simple. In [4] the zeros of f_κ in $\mathbb{C}_1^*$ are determined asymptotically as Re $(\kappa) \to \infty$ (even if κ runs through complex values such that Im $\kappa/(\text{Re}\,\kappa + 1)$ is constant). Based on this local result for the zeros the asymptotic distribution function was determined, when $0 < \kappa \to \infty$. Avoiding these local approximations of the zeros which require some voluminous analysis here we derive the limit distribution of the zeros via Lemma 2.

First we deal with the case $\kappa = n \in \mathbb{N}$, where the zeros x_{nv} of P_n are numbered according to (0.1) and its counting function is given by (0.2). We start with the Lindelöf–Wirtinger expansion for f_n [24, 4]

$$f_n(z) = \frac{P_n(z)}{(1-z)^{n+1}} = n! \sum_{m=-\infty}^{\infty} \frac{1}{(2\pi i m + \log(1/z))^{n+1}}, \qquad n \in \mathbb{N}, \quad (3.1)$$

valid in $\mathbb{C} - \{1\}$, where for the logarithm the principal branch in $\mathbb{C}_0^*$ is chosen; that is, $\log(1/z)$ is real for real positive z. Rewriting (3.1) as

$$\frac{P_n(z)}{n!} = \left(\frac{1-z}{\log(1/z)}\right)^{n+1} \left\{1 + (\log(1/z))^{n+1} \sum_{m \neq 0} \frac{1}{(2\pi i m + \log(1/z))^{n+1}}\right\}$$

we obtain for z in some neighbourhood of $z = 1$

$$\lim_{n \to \infty} \frac{1}{n} \frac{P_n'(z)}{P_n(z)} = g(z), \qquad (3.2)$$

where

$$g(z) := \frac{d}{dz} \log\left(\frac{1-z}{\log(1/z)}\right) = \frac{1}{z-1} - \frac{1}{z \log z}. \qquad (3.3)$$

Putting

$$Q_n(z) := P_n(-z) \qquad \text{and} \qquad h(z) := = -g(-z) \qquad (3.4)$$

we have

$$\lim_{n \to \infty} \frac{1}{n} \frac{Q_n'(z)}{Q_n(z)} = h(z) \qquad (3.2)'$$

for z in some neighbourhood of $z = -1$. From (3.3) and (3.4) we get $-xh(-x) \to 1$ as $0 < x \to \infty$, that is (1.7) holds. Thus Lemma 2 combined with (3.2)′ implies the existence of a unique $F \in M_+$ such that

$$h(z) = \int_0^\infty \frac{dF(t)}{z-t}, \qquad z \in \mathbb{C}_0^*.$$

Further, by the analytical properties of h (compare the proof of Theorem 1) we get

$$h(z) = \int_0^\infty \frac{f(t)}{z-t}\, dt, \qquad z \in \mathbb{C}_0^*,$$

where $f = F'$. As above, by Theorem 7b in [21, p. 340], we conclude

$$f(t) = \frac{1}{t(\pi^2 + \log^2 t)}, \qquad t > 0.$$

Combining these results we end with (observe (3.4))

THEOREM 5. *If P_n denotes the Jonquière polynomial of degree n and $N_n(\xi)$ is the number of zeros of P_n not exceeding ξ, $\xi \leqslant 0$, then*

$$\begin{aligned} \lim_{n \to \infty} \frac{1}{n} N_n(\xi) &= \int_{-\infty}^{\xi} \frac{dt}{(-t)(\pi^2 + \log^2(-t))} \\ &= 1 - \frac{1}{\pi} \operatorname{arc\,cotan}\left(\frac{1}{\pi} \log \frac{-1}{\xi}\right) \end{aligned}, \qquad \xi < 0, \qquad (3.5)$$

arc cotan being the branch with $0 < \operatorname{arc\,cotan} x < \pi$ for real x.

Remark. Since $z^n P_n(1/z) = P_n(z)$ [7, 14] we have for

$$x_{n1} < \cdots < x_{n,n-1} < x_{nn} = 0, \qquad x_{n\nu} = 1/x_{n\ \nu}, \ \nu = 1, ..., n-1.$$

This kind of symmetry is reflected by the relation $F(1/\xi) = 1 - F(\xi)$ where F denotes the limit distribution in (3.5).

Finally we extend Theorem 5 to f_κ (see (0.5)′) the zeros of which we may assume to be numbered according to Peyerimhoff's result [14] as

$$x_{n,1}(\kappa) < x_{n,2}(\kappa) < \cdots < x_{n,n-1}(\kappa) < x_{n,n}(\kappa) = 0, \qquad n-1 < \kappa \leqslant n, \ n \in \mathbb{N}.$$

It follows from the monotonicity of the zeros with respect to κ [23, observe the different numbering of the zeros] and the separating property of the zeros of f_n (see (0.5) and $zf_n'(z) = f_{n+1}(z)$) that we have

$$x_{n,\nu-1}(n) \leqslant x_{n-1,\nu-1}(n-1) \leqslant x_{n,\nu}(\kappa), \qquad \nu = 2, ..., n$$

and

$$x_{n\nu}(\kappa) \leqslant x_{n\nu}(n), \qquad \nu = 1, ..., n,$$

where $n-1 < \kappa \leqslant n$, $n \in \mathbb{N}$. Denoting as above

$$N_n(f_\kappa; \xi) = |\{\nu \in \{1, ..., n\} \mid x_{n\nu}(\kappa) \leqslant \xi\}|$$

for given $\xi \leqslant 0$ there exists a $\nu_0 \in \{1, \ldots, n\}$ such that $x_{n,\nu_0-1}(n) < \xi \leqslant x_{n\nu_0}(n)$ (put $x_{n,0} := -\infty$). Hence we get

$$\begin{aligned} N_n(f_\kappa; \xi) &= N_n(P_n; \xi) && \text{if } \xi < x_{n,\nu_0}(\kappa) \text{ or } \xi = x_{n,\nu_0}(n) \\ &= N_n(P_n; \xi) + 1 && \text{if } x_{n,\nu_0}(\kappa) \leqslant \xi < x_{n,\nu_0}(n) \end{aligned}$$

and so the desired extension of Theorem 5 by:

COROLLARY TO THEOREM 5. *If f_κ denotes Jonquière's function (defined in* (0.5)$'$), $n-1 < \kappa \leqslant n$, $n \in \mathbb{N}$, *and $N_n(\xi)$ is the number of zeros of f_κ not exceeding ξ, $\xi \leqslant 0$, then* (3.5) *holds.*

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